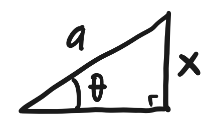
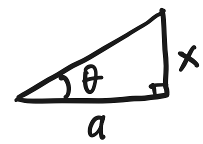
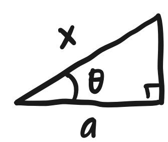
Trigonometric Substitution.

General Strategy: If we see the expression x^2+a^2, x^2-a^2, or a^2-x^2 in the integrand with $a>0$, we may be able to use a trig substitution to turn the integral into a trigonometric integral.

There are 3 choices for substitution.

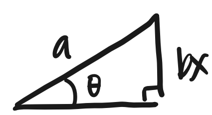
Case	Substitution	Pythagorean Identity	Right Triangle (SOH CAH TOA)
① a^2-x^2	$x=a\sin\theta$, $dx=a\cos\theta\,d\theta$	$a^2-(a\sin\theta)^2=(a\cos\theta)^2$ from $\sin^2\theta+\cos^2\theta=1$	$\sin\theta=\frac{x}{a}\,\frac{:\text{opp}}{:\text{hyp}}$ 
② x^2+a^2	$x=a\tan\theta$, $dx=a\sec^2\theta\,d\theta$	$(a\tan\theta)^2+a^2=(a\sec\theta)^2$ from $\tan^2\theta+1=\sec^2\theta$	$\tan\theta=\frac{x}{a}\,\frac{:\text{opp}}{:\text{adj}}$ 
③ x^2-a^2	$x=a\sec\theta$, $dx=a\sec\theta\tan\theta\,d\theta$	$(a\sec\theta)^2-a^2=(a\tan\theta)^2$ from $\tan^2\theta+1=\sec^2\theta$	$\sec\theta=\frac{x}{a}$, $\cos\theta=\frac{a}{x}\,\frac{:\text{adj}}{:\text{hyp}}$ 

Remarks:

★ The variable x can be replaced with a linear term bx with $b>0$. We also replace dx with $b\,dx$.

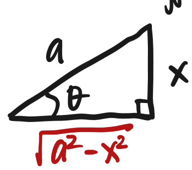
For example, if we have $a^2-b^2x^2=a^2-(bx)^2$, we use the substitution $bx=a\sin\theta$ and $b\,dx=a\cos\theta\,d\theta$;

We also have the right triangle by $\sin\theta=\frac{bx}{a}\,\frac{:\text{opp}}{:\text{hyp}}$

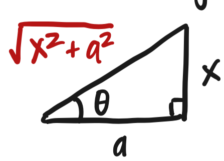


★ The missing sides of the right triangles can be found using the Pythagorean Theorem: $(\text{hyp})^2=(\text{opp})^2+(\text{adj})^2$

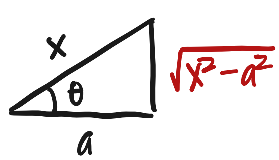
$\sin\theta=\frac{x}{a}\,\frac{:\text{opp}}{:\text{hyp}}$



$\tan\theta=\frac{x}{a}\,\frac{:\text{opp}}{:\text{adj}}$



$\cos\theta=\frac{a}{x}\,\frac{:\text{adj}}{:\text{hyp}}$



How do we use Trig Substitution?

1. Determine which trig substitution ($x = a\sin\theta$, $x = a\tan\theta$, or $x = a\sec\theta$) is appropriate.
2. Use the substitution to get an integral in terms of θ.
   Don't forget to substitute dx with ($dx = a\cos\theta\, d\theta$, $dx = a\sec^2\theta\, d\theta$, or $dx = a\sec\theta\tan\theta\, d\theta$)!
   The integral we end up with should be a trig. integral.
3. Find the antiderivative of the trig integral! This is still in terms of θ.
4. Use the right triangles + SOH CAH TOA to determine an expression of antiderivative in terms of x!
   If θ is present in the antiderivative (not inside a trig. function),
   use the corresponding arc/inverse function: ($\theta = \arcsin\left(\frac{x}{a}\right)$, $\theta = \arctan\left(\frac{x}{a}\right)$, or $\arccos\left(\frac{a}{x}\right)$).

Remarks:

✱ **Power Reduction Identities** are useful for taking antiderivatives.

$$\boxed{\sin^2(\theta) = \frac{1}{2}(1-\cos(2\theta))} \quad \text{and} \quad \boxed{\cos^2(\theta) = \frac{1}{2}(1+\cos(2\theta))}$$

✱ When evaluating inv. trig functions or translating x-values to θ-values using a trig. sub, remember that the output angles / θ-values are restricted on specific intervals.

Inverse Trig Function	The inputs/arguments on the inv. trig functions	The values of θ are restricted on these intervals!
1. $\theta = \arcsin(x)$	$x \in [-1,1]$	$\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
2. $\theta = \arctan(x)$	$x \in (-\infty, \infty)$	$\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
3. $\theta = \arccos(x)$	$x \in [-1,1]$	$\theta \in [0, \pi]$
$\theta = \text{arcsec}(x)$	$x \in [1, \infty)$	$\theta \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

✱ **Double angle identities** are useful for evaluating antiderivatives when power reduction is used.

$$\sin(2\theta) = 2\sin\theta\cos\theta$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta = \frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$\tan(2\theta) = \frac{2\tan\theta}{1-2\tan^2\theta}$$

Examples.

① $I_1 = \int \frac{3}{4+x^2}\,dx \overset{\text{trigsub}}{=} 3\int \frac{1}{4\sec^2\theta} 2\sec^2\theta\,d\theta = \frac{3}{2}\int 1\,d\theta = \frac{3}{2}\theta + C \overset{\text{trigsub}}{=} \frac{3}{2}\arctan\left(\frac{x}{2}\right) + C$.

Trig Sub: $x = 2\tan\theta,\ dx = 2\sec^2\theta\,d\theta$: $\tan\theta = \frac{x}{2},\ \theta = \arctan\left(\frac{x}{2}\right)$

$4 + x^2 = 4 + 4\tan^2\theta = 4\sec^2\theta$

Tip: Break down your calculation / Do it in steps to organize your work.

② $I_2 = \int \frac{1}{\sqrt{4-x^2}}\,dx \overset{\text{trigsub}}{=} \int \frac{1}{2\cos\theta} 2\cos\theta\,d\theta = \int 1\,d\theta = \theta + C \overset{\text{trigsub}}{=} \arcsin\left(\frac{x}{2}\right) + C$.

Trig Sub: $x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta$; $\sin\theta = \frac{x}{2},\ \theta = \arcsin\left(\frac{x}{2}\right)$;

$4 - x^2 = 4 - (2\sin\theta)^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$; $\sqrt{4\cos^2\theta} = 2\cos\theta$;

③ $I_3 = \int \sqrt{4-x^2}\,dx \overset{\text{trigsub}}{=} \int 2\cos\theta\,(2\cos\theta)\,d\theta = \int 4\cos^2\theta\,d\theta \overset{\text{power red.}}{=} \int (4)\left(\frac{1}{2}\right)(1+\cos(2\theta))\,d\theta$

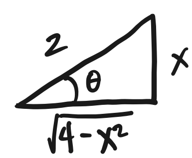
Trig Sub: $x = 2\sin\theta,\ dx = 2\cos\theta\,d\theta$;

$4 - x^2 = 4 - 4\sin^2\theta = 4\cos^2\theta$; $\sqrt{4-x^2} = \sqrt{4\cos^2\theta} = 2\cos\theta$

$= \int (2 + 2\cos(2\theta))\,d\theta = 2\theta + 2\left(\frac{1}{2}\right)\sin(2\theta) + C \overset{\text{double angle}}{=} 2\theta + 2\sin\theta\cos\theta + C$

From Trig Sub: $\sin\theta = \frac{x}{2} \begin{array}{l}: \text{opp}\\ : \text{hyp}\end{array}$

$\theta = \arcsin\left(\frac{x}{2}\right)$



$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{4-x^2}}{2}$

$= 2\arcsin\left(\frac{x}{2}\right) + 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C = 2\arcsin\left(\frac{x}{2}\right) + \frac{1}{2}x\sqrt{4-x^2} + C$;

④ $I_4 = \int \frac{1}{(1+x^2)^2}\,dx \overset{\text{TrigSub}}{=} \int \frac{1}{(\sec^2\theta)^2}\sec^2\theta\,d\theta = \int \cos^2\theta\,d\theta \overset{\text{trig id.}}{=} \int \frac{1}{2}(1+\cos(2\theta))\,d\theta$

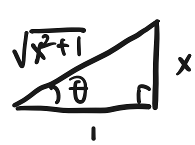
TrigSub: $x = \tan\theta,\ dx = \sec^2\theta\,d\theta$;

$1 + x^2 = 1 + \tan^2\theta = \sec^2\theta$;

$= \frac{1}{2}\theta + \frac{1}{2}\sin(2\theta)\cdot\frac{1}{2} + C \overset{\text{trig id}}{=} \frac{1}{2}\theta + \frac{1}{2}\sin\theta\cos\theta + C$

From Trig Sub: $\tan\theta = \frac{x}{1} \begin{array}{l}: \text{opp}\\ : \text{adj}\end{array}$

$\theta = \arctan(x)$



$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{1}{\sqrt{x^2+1}}$, $\sin\theta = \frac{\text{opp}}{\text{hyp}} = \frac{x}{\sqrt{x^2+1}}$

$= \frac{1}{2}\arctan(x) + \frac{1}{2}\left(\frac{x}{\sqrt{x^2+1}}\right)\left(\frac{1}{\sqrt{x^2+1}}\right) + C = \frac{1}{2}\arctan(x) + \frac{x}{2(x^2+1)} + C$;

⑤ $I_5 = \int \frac{1}{a^2+b^2x^2}\,dx$ for constants $a,b>0$.

Trig Sub: $bx = a\tan\theta$, $b\,dx = a\sec^2\theta\,d\theta$; $dx = \frac{a}{b}\sec^2\theta\,d\theta$

$a^2+b^2x^2 = a^2+(bx)^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$

$$I_5 \overset{\text{trig sub}}{=} \int \frac{1}{a^2\sec^2\theta}\left(\frac{a}{b}\right)\sec^2\theta\,d\theta = \frac{1}{ab}\int 1\,d\theta = \frac{1}{ab}\theta + C \overset{\text{trig sub}}{=} \frac{1}{ab}\arctan\left(\frac{b}{a}x\right) + C \;;$$

from Trig Sub: $bx = a\tan\theta$, $\tan\theta = \frac{b}{a}x$, $\theta = \arctan(\frac{b}{a}x)$

⑥ $I_6 = \int \frac{x}{a^2+b^2x^2}\,dx$ for constants $a,b>0$

Method 1: U-substitution (preferred)

$$I_6 = \int \frac{x}{a^2+b^2x^2}\,dx \overset{u\text{-sub}}{=} \int \left(\frac{1}{2b^2}\right)\frac{1}{u}\,du = \frac{1}{2b^2}\ln|u| + C \overset{u\text{-sub}}{=} \frac{1}{2b^2}\ln|a^2+b^2x^2| + C \;;$$

u-sub: $u = a^2+b^2x^2$, $du = 2b^2x\,dx$, $x\,dx = \frac{1}{2b^2}du$

Method 2: Trig Sub (here for comparison)

Trig Sub: $bx = a\tan\theta$, $b\,dx = a\sec^2\theta\,d\theta$, $dx = \frac{a}{b}\sec^2\theta\,d\theta$

$a^2+b^2x^2 = a^2 + a^2\tan^2\theta = a^2\sec^2\theta$

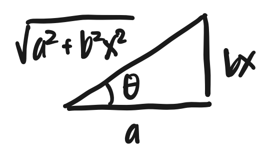
$$I_6 \overset{\text{trig sub}}{=} \int \left(\frac{a}{b}\tan\theta\right)\frac{1}{\cancel{a^2\sec^2\theta}}\left(\frac{a}{b}\right)\cancel{\sec^2\theta}\,d\theta = \frac{1}{b^2}\int \tan\theta\,d\theta = \frac{1}{b^2}\int\frac{\sin\theta}{\cos\theta}\,d\theta \overset{u\text{-sub}}{=} \frac{1}{b^2}\int\frac{1}{u}(-1)\,du$$

u-sub: $u = \cos\theta$, $du = -\sin\theta\,d\theta$

$$= -\frac{1}{b^2}\ln|u| + C \overset{u\text{-sub}}{=} -\frac{1}{b^2}\ln|\cos\theta| + C \overset{\text{trig sub}}{=} -\frac{1}{b^2}\ln\left|a(a^2+b^2x^2)^{-\frac{1}{2}}\right| + C$$

from Trig Sub: $bx = a\tan\theta$

$\tan\theta = \frac{bx}{a} \begin{matrix}: \text{opp} \\ : \text{adj}\end{matrix}$



$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{a}{\sqrt{a^2+b^2x^2}} = a(a^2+b^2x^2)^{-\frac{1}{2}}$

$$= \underbrace{-\frac{1}{b^2}\ln(a)}_{\text{constant, absorb into C}} + \frac{1}{2b^2}\ln|a^2+b^2x^2| + C = \frac{1}{2b^2}\ln|a^2+b^2x^2| + C \;;$$

⑦ $$I_7 = \int\frac{\sqrt{x^2-25}}{x}\,dx \overset{\text{trig sub}}{=} \int\frac{5\tan\theta}{\cancel{5\sec\theta}}(5)\cancel{\sec\theta}\tan\theta\,d\theta = 5\int\tan^2\theta\,d\theta = 5\int(\sec^2\theta - 1)\,d\theta$$

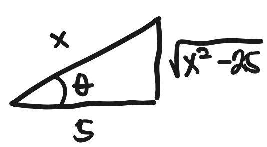
Trig Sub: $x = 5\sec\theta$, $dx = 5\sec\theta\tan\theta\,d\theta$

$x^2-25 = 25\sec^2\theta - 25 = 25\tan^2\theta$, $\sqrt{x^2-25} = 5\tan\theta$

$$= 5\tan\theta - 5\theta + C = 5\left(\frac{\sqrt{x^2-25}}{5}\right) - 5\arccos\left(\frac{5}{x}\right) + C = \sqrt{x^2-25} - 5\arccos\left(\frac{5}{x}\right) + C \;;$$

from Trig Sub: $\sec\theta = \frac{x}{5}$, $\cos\theta = \frac{5}{x} \begin{matrix}: \text{adj} \\ : \text{hyp}\end{matrix}$

$\theta = \arccos(\frac{5}{x})$



$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{\sqrt{x^2-25}}{5}$

Examples with Evaluation/Definite Integrals.

① $\int_0^3 \sqrt{9-x^2}\,dx$

Method 1. Find indefinite integral. Then, evaluate in terms of x

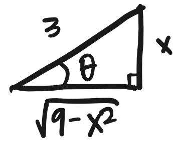
$$F(x) = \int \sqrt{9-x^2}\,dx \overset{\text{Trig Sub}}{=} \int 3\cos\theta\,(3\cos\theta)\,d\theta = \int \frac{9}{2}\left(1+\cos(2\theta)\right)d\theta = \frac{9}{2}\theta + \frac{9}{2}\left(\frac{1}{2}\right)\sin(2\theta) + C$$

Trig Sub: $x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta$

$9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta\ ;\ \sqrt{9-x^2} = 3\cos\theta$

$$= \frac{9}{2}\theta + \frac{9}{2}\sin\theta\cos\theta + C = \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{9}{2}\left(\frac{x}{3}\right)\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

From Trig Sub: $\sin\theta = \frac{x}{3} \begin{array}{l} :\text{opp} \\ :\text{hyp}\end{array}$



$\cos\theta = \frac{\text{adj}}{\text{hyp}} = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\arcsin\left(\frac{x}{3}\right) + \frac{1}{2}x\sqrt{9-x^2} + C\ ;$$

arcsin(x) is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$$\int_0^3 \sqrt{9-x^2}\,dx = F(3) - F(0) = \left[\frac{9}{2}\arcsin\left(\frac{3}{3}\right) - \cancel{\frac{1}{2}(3)\sqrt{9-(3)^2}}^{\,0}\right] - \left[\cancel{\frac{9}{2}\arcsin\left(\frac{0}{3}\right)}^{\,0} - \cancel{\frac{1}{2}(0)\sqrt{9-0^2}}^{\,0}\right]$$

$$= \frac{9}{2}\arcsin(1) = \frac{9}{2}\left(\frac{\pi}{2}\right) = \frac{9\pi}{4}\ ;$$

Method 2. Convert x-values to θ-values first and evaluate the antiderivative in terms of θ.

$$\int_0^3 \sqrt{9-x^2}\,dx \overset{\text{Trig Sub}}{=} \int_0^{\frac{\pi}{2}} 3\cos\theta\,(3\cos\theta)\,d\theta = \int_0^{\frac{\pi}{2}} 9\left(\frac{1}{2}\right)\left(1+\cos(2\theta)\right)d\theta = \frac{9}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\frac{\pi}{2}}$$

Trig Sub: $x = 3\sin\theta,\ dx = 3\cos\theta\,d\theta$

$\theta = \arcsin(\frac{1}{3}x)\ ;\quad x = 3:\ \theta = \arcsin(1) = \frac{\pi}{2}$

$x = 0:\ \theta = \arcsin(0) = 0$ ← arcsin(x) is restricted to $[-\frac{\pi}{2}, \frac{\pi}{2}]$

$9 - x^2 = 9 - 9\sin^2\theta = 9\cos^2\theta\ ;\ \sqrt{9-x^2} = 3\cos\theta\ ;$

$$= \frac{9}{2}\left[\frac{\pi}{2} + \cancel{\frac{1}{2}\sin\left(2\cdot\frac{\pi}{2}\right)}^{\,0}\right] - \cancel{\frac{9}{2}\left[0 + \frac{1}{2}\sin(2\cdot 0)\right]}^{\,0} = \frac{9\pi}{4}\ ;$$